University of California, Berkeley
Department of Mechanical Engineering
ME 170, Spring 2017

Homework 6

Problem 1

Let a simple pendulum be modeled as a particle P of mass m attached by a massless rigid rod OP of length ℓ to a pivot O which is fixed in an inertial frame of reference. Recall that

$$\ddot{\theta} + \omega_n^2 \sin\theta = 0, \qquad \omega_n^2 = \frac{g}{\ell}, \tag{1}$$

and

$$\frac{\dot{\theta}^2}{\omega_n^2} + 2(1 - \cos\theta) = 4\,\frac{E}{E^*}, \tag{2}$$

where $E^* = 2mg\ell$.

(a) Plot the phase portrait of the simple pendulum in the plane $x = \theta$, $y = \dfrac{\dot{\theta}}{\omega_n}$ for

$$\frac{E}{E^*} = 0.25, 0.50, 0.75, 1.00, 1.25, 1.50, 1.75, 2.00.$$

Use the same scale on both axes.

(b) If the initial conditions lie anywhere inside a circle of radius 0.1 centered at the origin in phase space, calculate the radius of the corresponding accessible region.

(c) Use the result of Part (b) as the basis for an argument to prove the Lyapunov stability of the lower equilibrium point.

(d) By Lagrange's stability theorem, if $V(\theta)$ has a strict minimum at an equilibrium position, then that position is Lyapunov stable. Noting that

$$\frac{dV}{d\theta} = 0, \quad \frac{d^2V}{d\theta^2} > 0, \tag{3}$$

deduce that the position $\theta = 0$ of the pendulum is a stable equilibrium position.

(e) Writing Eq. (1) in the phase plane form

$$\begin{aligned} \dot{x} &= \omega_n y \\ \dot{y} &= -\omega_n \sin x, \end{aligned} \tag{4}$$

(with $x = \theta$), use the condition

$$0 = \text{div}\mathbf{V} = \frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y} \tag{5}$$

to show that the phase flow is area-preserving.

(f) Consider the function

$$\Lambda(x, y) = 4\frac{E}{E^*} \tag{6}$$

for the simple pendulum. Show that

$$\Lambda(0,0) = 0, \quad \Lambda(x,y) > 0 \quad \text{for} \quad (x,y) \neq (0,0), \tag{7}$$

and

$$\dot{\Lambda}(x,y) \leq 0 \tag{8}$$

for all (x, y). ($\dot{\Lambda}(x,y)$ is automatically zero at $(0,0)$ since $(\dot{x}, \dot{y}) = (0,0)$ there.) Apply Lyapunov's stability theorem, i.e., show that $(0,0)$ is a Lyapunov stable equilibrium position.

Problem 2

Suppose that the simple pendulum in Problem 1 is subject to a drag force of the form

$$\mathbf{D} = -c\,\ell\,\dot{\theta}\,\mathbf{e}_\theta \qquad (c > 0). \tag{9}$$

(a) Obtain the governing equation for θ.

(b) Apply the power theorem, namely

$$\mathbf{F} \cdot \mathbf{v} = \dot{T}, \tag{10}$$

to prove that the total mechanical energy E satisfies the inequality

$$\dot{E} \leq 0 \tag{11}$$

for every motion of the damped pendulum.

(c) Argue that the lower equilibrium point is asymptotically stable.

(d) Is E a Lyapunov function for the damped pendulum? Is it a strict Lyapunov function?

Problem 3

In this problem, we study linearization of the differential equation (1) about the lower and upper equilibrium points. Recall Taylor's theorem

$$f(\theta) = f(\theta_0) + f'(\theta_0)(\theta - \theta_0) + \frac{1}{2}f''(\theta_0)(\theta - \theta_0)^2 + \ldots \tag{12}$$

(a) Take $\theta_0 = 0$, and argue that at least for t close to zero, Eq. (1) may be approximated by the linear equation

$$\ddot{\theta} + \omega_n^2 \theta = 0. \tag{13}$$

(b) For small initial values of θ and $\dot{\theta}/\omega_n$, discuss the motion of the pendulum as predicted by Eq. (13). Does Eq. (13) continue to hold for all values of t? Why?

(c) Obtain the corresponding approximation to the energy equation (2). Sketch the approximate phase portrait in the vicinity of the origin in phase space.

(d) Take $\theta_0 = \pi$ and introduce the variable $\xi = \theta - \pi$. Show that Eq. (1) may be linearized as

$$\ddot{\xi} - \omega_n^2 \xi = 0. \tag{14}$$

(e) Argue that

$$\xi = A \sinh \omega_n t + B \cosh \omega_n t \tag{15}$$

is the general solution to (14) and relate the constants A and B to the initial small values of ξ and $\dot{\xi}$.

(f) Approximate the energy equation (2) for small values of ξ and $\dot{\xi}$. Sketch the approximate phase portrait in the vicinity of the point $\theta = \pi$.

(g) What does the approximate phase portrait in Part (f) indicate regarding the stability of the upper equilibrium point of the pendulum?

(h) If at time $t = 0$, $\xi = 0$, and $\dot{\xi} = \delta$, an arbitrary small positive number, solve (14) for the predicted motion. Argue that the approximation (14) will break down.

Problem 4

Suppose that a particle P of mass m is moving rectilinearly and has a potential energy

$$V(x) = \frac{1}{2}kx^2 - \frac{1}{3}bx^3 \quad \text{J}, \tag{16}$$

where k and b are positive constants.

(a) Calculate the force that acts on P and write the governing equations for x.

(b) Determine the equilibrium positions.

(c) Calculate the second derivative of V at the equilibrium positions.

(d) Sketch the graph of $V(x)$.

(e) Write out the equation for conservation of energy for P.

(f) Show that the value of total energy which corresponds to the separatrix is $E = \frac{1}{6}\frac{k^3}{b^2}$, and obtain the equation of the separatrix.

(g) Sketch or plot the phase portrait.

(h) Discuss the stability of the equilibrium points.

(i) Show that Lagrange's equation holds in the form

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0, \tag{17}$$

where $L(x, \dot{x}) = T(\dot{x}) - V(x)$.

Problem 5

Let a narrow rigid tube be modeled as a circle of radius r, and suppose that is rotating about its vertical diameter with constant angular velocity $\dot{\theta} = \Omega$. Also, suppose that a particle P of mass m can slide inside the tube. Neglect friction. Gravity acts in the $-\mathbf{k}$ direction. The reaction force exerted on P by the tube may be represented in the spherical coordinate system as $\mathbf{N} = N_r\mathbf{e}_r + N_\theta\mathbf{e}_\theta$.

(a) Write the component equations of Newton's second law for P.

(b) Deduce that

$$\ddot{\phi} = \left(\frac{g}{r} + \Omega^2\cos\phi\right)\sin\phi. \tag{18}$$

(c) Show that the kinetic energy can be written in the form

$$T = T_2 + T_1 + T_0, \tag{19}$$

where T_2 is quadratic in $\dot{\phi}$, T_1 is linear in $\dot{\phi}$, and T_0 is a function of ϕ only. (In this problem, T_1 is actually zero.)

(d) Apply the work-energy theorem, i.e.,

$$P = \dot{T}. \tag{20}$$

Write the power due to gravity as $-\dot{V}$. Show that there is an energy integral of the form

$$E' = T_2 - T_0 + V = \text{const.} \tag{21}$$

(e) Differentiate (21) with respect to t and compare with (18).

(f) Show that the power of the reaction force is $2\dot{T}_0$.

(g) Show that there are two solutions to Eq. (18) in which the particle is at rest.

(h) Look for a steady (non-equilibrium) solution $\phi(t) = \text{const.} = \phi_0$. Calculate N_θ.

(i) Consider solutions of the form $\phi(t) = \phi_0 + \xi(t)$, where $\xi(t)$ are small perturbations. Linearize the differential equation (18) about the steady solution. Obtain the differential equation for ξ. What is the form of the solutions $\xi(t)$?